

Levinson Theorem for Differential Equations with Piecewise Constant Argument Generalized

Samuel Castillo*

Departamento de Matemática. Facultad de Ciencias.
Universidad del Bío-Bío. Casilla 5-C. Concepción. Chile.
scastill@ubiobio.cl

Wilfred Flores

Facultad de Ingeniería. Universidad de Talca.
Campus Curicó. Camino Los Niches Km 1, Curicó. Chile.
wflores51@gmail.com

Abstract

In this work, it is presented an adaptation of an asymptotic theorem of N. Levinson of 1948, to differential equation with piecewise constant argument generalized, which were introduced by M. Akhmet in 2007. The N. Levinson's theorem which is adapted is that dealt by M. S. P. Eastham in his work which is present in this bibliography. The more relevant hypotheses of this theorem are highlighted and it is established a version of this theorem with these hypotheses for ordinary differential equations. Such a version is that which is adapted to differential equation with piecewise constant argument generalized. The adaptation is proved by mean the Banach fixed point where contractive operator is built from a suitable version of the constant variation formula.

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1 Introduction

Let $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ the sets of the positive integer numbers, of the nonnegative integer numbers, of the integer numbers, of the real numbers, of the complex numbers, respectively. Let $N \in \mathbb{N}$. We will consider $\mathbb{K}^N := \mathcal{M}_{N \times 1}(\mathbb{K})$, i.e, as column vectors, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We will denote by $|\cdot|$ to the Euclidean norm for \mathbb{K}^N . $\mathcal{M}_N(\mathbb{C})$ will denote the $N \times N$ matrix with complex entries.

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On $\mathcal{M}_N(\mathbb{C})$, $\|\cdot\|$ will denote the classic norm operator which is defined for $A \in \mathcal{M}_N(\mathbb{C})$, by $\|A\| = \sup_{v \in \mathbb{C}^N - \{0\}} \frac{|Av|}{|v|}$.

A Differential Equations with Piecewise Constant Argument Generalized (DEPCAG) is a differential equation of the form

$$\frac{dx}{dt} = f(t, x(t), x(\gamma(t))), \quad (1)$$

where $\gamma : [t_0, +\infty[\rightarrow [t_0, +\infty[$ is such that there is an strictly increasing sequence $(t_n)_{n=0}^{+\infty}$ with $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$ and $\gamma([t_n, t_{n+1}[) = \{\xi_n\}$ for all $n \in \mathbb{N}$. A function $x = x(t)$ is understood to be *solution of the DEPCAG (1)* if:

1. x is continuous on $[t_0, +\infty[$;
2. the derivative $\frac{dx}{dt}$ of x with the possible exception in $t = t_n$ for $n \in \mathbb{N}_0$, where is unilateral derivative exists;
3. x is a solution of (1) with the possible exception in $t = t_n$ for all $n \in \mathbb{N}_0$.

Notice that (1) is an ordinary differential equation in each interval $[t_n, t_{n+1}[$ for all $n \in \mathbb{N}_0$, but the leaps between those intervals creates a difference system of the form

$$\begin{aligned} x(\xi_n) &= x(t_n) + \int_{t_n}^{\xi_n} f(\zeta, x(\zeta), x(\xi_n)) d\zeta, \\ x(t_{n+1}) &= x(\xi_n) + \int_{\xi_n}^{t_{n+1}} f(\zeta, x(\zeta), x(\xi_n)) d\zeta, \end{aligned}$$

for all $n \in \mathbb{N}_0$.

The name generalized in those equations, is explained by the inclusion of the differential equations with piecewise argument whose study seems to be started by K. Cooke and J. Wiener in [6, 11, 12]. They consider equations of the form 1 with $\gamma(t) = [t]$ or $\gamma(t) = 2 \left\lfloor \frac{t+1}{2} \right\rfloor$, where $[\cdot]$ is the function assigns to each real number, the greater integer number less than it. The first known generalization was made by M. Akhmet [1].

In this work we establish a version of the asymptotic Levinson's theorem N. Levinson [8, (1948)] (see Eastham [7]) for DEPCAG

$$\frac{dy}{dt} = A(t)y(t) + B(t)y(\gamma(t)) + F(t, z(g(t))), \quad t \in [t_0, +\infty[, \quad (2)$$

where $A(t)$, $B(t)$ are matrices in $\mathcal{M}_N(\mathbb{C})$, whose coefficients are locally integrable functions of t , $F : [t_0, +\infty[\times \mathbb{C}^N \rightarrow \mathbb{C}^N$ is a locally integrable function in the first variable such that there is $\eta : [t_0, +\infty[\rightarrow \mathbb{R}_0^+$ such that

$$\begin{aligned} |F(t, \hat{a}) - F(t, \hat{b})| &\leq \eta(t) |\hat{a} - \hat{b}| \\ F(t, 0) &= 0, \end{aligned} \quad (3)$$

$\eta(t)$ will satisfy an L^1 type condition which will be given below (see (23)), γ is a piecewise constant argument defined as above and $g : [t_0, +\infty[\rightarrow [t_0, +\infty[$ satisfy

$$g([t_n, t_{n+1}[) \subseteq [t_n, t_{n+1}[,$$

for all $n \in \mathbb{N}_0$.

Notice that g could be an piecewise constant argument.

DEPCAG (2) will be seen as a perturbation of the DEPCAG

$$\frac{dz}{dt} = A(t)z + B(t)z(\gamma(t)) \quad (4)$$

which will have dichotomic condition similar to those given in the asymptotic Levinson theorem considered here.

An asymptotic result for DEPCAG is given by M. Akhmet [2]. He considers the DEPCA,

$$\frac{dy}{dt} = C_0 y + f(t, x(t), x(\gamma(t))), \quad (5)$$

as a perturbation of the autonomous ordinary differential equation

$$\frac{dx}{dt} = C_0 x, \quad (6)$$

where γ is defined as above and the following hypotheses are given

$$\exists L > 0 : \|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq L(\|x_2 - x_1\| + \|y_2 - y_1\|) \text{ y } f(t, 0, 0) = 0; \quad (7a)$$

$$\exists \bar{t} : 0 < t_{n+1} - t_n \leq \bar{t}; \quad (7b)$$

$$\begin{aligned} \exists M, m > 0 : m \leq \|e^{C(t-s)}\| \leq M, \forall t, s \in [t_n, t_{n+1}]; ML\bar{t}e^{ML\bar{t}} < 1; \\ 2ML\bar{t} < 1; M^2\bar{L}\bar{t} \left(\frac{ML\bar{t}e^{ML\bar{t}} + 1}{1 - ML\bar{t}e^{ML\bar{t}}} + ML\bar{t}e^{ML\bar{t}} \right) < m; \end{aligned} \quad (7c)$$

$$\exists \eta : \mathbb{R}^+ \rightarrow [0, L] : \|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq \eta(t)(\|x_2 - x_1\| + \|y_2 - y_1\|); \quad (7d)$$

$$\ell_0 := \int_0^{+\infty} t^{m_\beta + m_\alpha - 2} e^{t(\beta - \alpha)} \eta(t) dt < +\infty, \quad (7e)$$

where $\lambda_1, \dots, \lambda_p$ are the characteristic values of C_0 , $\alpha = \min_{j=1, \dots, p} \operatorname{Re}(\lambda_j)$, $\beta = \max_{j=1, \dots, p} \operatorname{Re}(\lambda_j)$, m_α and m_β the maximum orders of the characteristic values of C with real part equal to α and β respectively, for all $t \in \mathbb{R}^+$.

Then, Akhmet [2] provides the following result.

Theorem 1 *Assume that (7a)-(7e) hold. Then, for every solution $y = y(t)$ of the DEPCAG (5) has a representation*

$$y(t) = e^{C_0 t} [c + w(t)], \quad (8)$$

where $w(t) \rightarrow 0$ as $t \rightarrow +\infty$ and $c \in \mathbb{R}^N$.

That result, consider a DEPCAG as a perturbation of an autonomous ordinary differential equation (6). In Theorem 1, it is seen the perturbation of the whole fundamental matrix of (6). In our result we see the perturbation of only one dimension of the solution space of (4), although our non perturbed equation is already a DEPCAG. with the elements for our proof The last section is devoted to the main result and its proof.

2 Preliminaries

In this section sets the definitions and facts to present the main result.

First we find the Cauchy matrix for system (4).

We ask the condition

$$D_n(t) = I + \int_{\xi_n}^t X(\xi_n, u)B(u)du \text{ is invertible} \quad (9a)$$

for all $t \in [t_n, t_{n+1}]$ and $n \in \mathbb{N}_0$, where $X(t, s) = X(t)X(s)^{-1}$ and X is a fundamental matrix for the system

$$\frac{dx}{dt} = A(t)x. \quad (10)$$

Let

$$\Phi(n) = H(n-1)H(n-2) \cdots H(0),$$

where

$$H(n) = X(t_{n+1}, \xi_n)D_n(t_{n+1})D_n(t_n)^{-1}X(\xi_n, t_n). \quad (11)$$

for all $n \in \mathbb{N}_0$ and for all $t \in [t_n, t_{n+1}]$.

Given $s, t \in [t_0, +\infty[$, we have that if $Z : [t_0, +\infty[\rightarrow \mathcal{M}_N(\mathbb{C})$ is a matrix function such that $Z(s, s) = I$ and for all $z_0 \in \mathbb{C}^N$, $z = Z(\cdot, s)z_0$ is a solution of (4) in the sense given in the introduction for equation (1) such that $z(s) = z_0$, then

$$\begin{aligned} Z(t, s) &= X(t, \gamma(t))D_{k_t}(t)D_{k_t}(t_{k_t})^{-1}X(\gamma(t), t_{k_t}) \\ &\times \Phi(k_t)\Phi(k_s+1)^{-1} \\ &\times X(t_{k_s+1}, \gamma(s))D_{k_s}(t_{k_s+1})D_{k_s}(s)^{-1}X(\gamma(s), s). \end{aligned} \quad (12)$$

Assume that there are an unitary vector \hat{e} and locally integrable functions $\hat{\lambda} : [t_0, +\infty[\rightarrow \mathbb{C}$ and $\hat{\lambda}_d : [t_0, +\infty[\rightarrow \mathbb{C}$ such that

$$Z(t, s)\hat{e} = \tilde{e}(t, s)\hat{e}, \quad (13a)$$

where

$$\begin{aligned}
\tilde{e}(t, s) &= e^{\int_s^t \hat{\lambda}(\xi) d\xi} \\
&\times \left(1 + \int_{\gamma(t)}^t e^{\int_{\gamma(t)}^u \hat{\lambda}(\xi) d\xi} \hat{\lambda}_d(u) du \right) \left(1 + \int_{\gamma(t)}^{t_{k_t}} e^{\int_{\gamma(t)}^u \hat{\lambda}(\xi) d\xi} \hat{\lambda}_d(u) du \right)^{-1} \\
&\times \left(\prod_{j=k_s+1}^{k_t} \left[1 + \int_{\xi_j}^{t_{j+1}} e^{\int_{\xi_j}^u \hat{\lambda}(\xi) d\xi} \hat{\lambda}_d(u) du \right] \left[1 + \int_{\xi_j}^{t_j} e^{\int_{\xi_j}^u \hat{\lambda}(\xi) d\xi} \hat{\lambda}_d(u) du \right]^{-1} \right) \\
&\times \left(1 + \int_{\gamma(s)}^{t_{k_s+1}} e^{\int_{\gamma(s)}^u \hat{\lambda}(\xi) d\xi} \hat{\lambda}_d(u) du \right) \left(1 + \int_{\gamma(s)}^s e^{\int_{\gamma(s)}^u \hat{\lambda}(\xi) d\xi} \hat{\lambda}_d(u) du \right)^{-1}
\end{aligned} \tag{13b}$$

and $k_t \in \mathbb{N}_0$ is defined for all $t \in [t_0, +\infty[$ such that $t \in [t_{k_t}, t_{k_t+1}[$.

Condition (9a) implies the invertibility of $Z(t, s)$ which allows to define $Z(t, s) = Z(s, t)^{-1}$ for $t < s$.

Now, we use the variation of constants formula for the DEPCAG

$$\frac{d\psi}{dt} = A(t)\psi(t) + B(t)\psi(\gamma(t)) + f(t). \tag{14}$$

It can be written as,

$$\psi(t) = Z(t, \gamma(t))\psi(\gamma(t)) + \int_{\gamma(t)}^t X(t, s)f(s)ds.$$

and

$$\psi(\gamma(t)) = Z(\gamma(t), t_{k_t})\psi(t_{k_t}) + \int_{t_{k_t}}^{\gamma(t)} X(\gamma(t), s)f(s)ds.$$

Then,

$$\psi(t) = Z(t, \gamma(t)) \left[Z(\gamma(t), t_{k_t})\psi(t_{k_t}) + \int_{t_{k_t}}^{\gamma(t)} X(\gamma(t), s)f(s)ds \right] + \int_{\gamma(t)}^t X(t, s)f(s)ds,$$

i.e.,

$$\psi(t) = Z(t, t_{k_t})\psi(t_{k_t}) + \int_{t_{k_t}}^t \Gamma(t, s)f(s)ds.$$

So,

$$\psi(t_{n+1}) = H(n)x(t_n) + \int_{t_n}^{t_{n+1}} \Gamma(t_{n+1}, s)f(s)ds,$$

for all $n \in [n_0, +\infty[\cap \mathbb{Z}$, where

$$\Gamma(t, s) = \begin{cases} Z(t, \gamma(t))X(t, s) & \text{if } t_{k_t} \leq s \leq \gamma(t) \\ X(t, s) & \text{if } \gamma(t) \leq t \leq t_{k_t+1}, \end{cases} \tag{15}$$

$s \in [t_n, t_{n+1}]$, $t \geq t_{n_0}$ and $n \in [n_0, +\infty[\cap \mathbb{Z}$.
So,

$$\begin{aligned}\psi(t) &= Z(t, t_{k_t}) [\Phi(k_t) \Phi(n_0)^{-1} \psi(t_{n_0}) \\ &+ \sum_{j=n_0}^{k_t-1} \Phi(k_t) \Phi(j+1)^{-1} \int_{t_j}^{t_{j+1}} \Gamma(t_{j+1}, s) f(s) ds \\ &+ \int_{t_{k_t}}^t \Gamma(t, s) f(s) ds.\end{aligned}$$

Hence,

$$\begin{aligned}\psi(t) &= Z(t, t_{n_0}) \psi(t_{n_0}) \\ &+ \sum_{j=n_0}^{k_t-1} \int_{t_j}^{t_{j+1}} Z(t, t_{j+1}) \Gamma(t_{j+1}, s) f(s) ds \\ &+ \int_{t_{k_t}}^t \Gamma(t, s) f(s) ds.\end{aligned}\tag{16}$$

Then, the integral equation (16) can be written as

$$\psi(t) = Z(t, t_{n_0}) \psi(t_{n_0}) + \int_{t_0}^t \hat{Z}(t, s) f(s) ds,\tag{17}$$

for $t \geq n_0$, where

$$\hat{Z}(t, s) = \begin{cases} Z(t, t_{n+1}) \Gamma(t_{n+1}, s) & \text{if } t_n \leq s \leq t_{n+1} \\ \Gamma(t, s) & \text{if } t_n \leq s \leq t_{n+1}, \end{cases}\tag{18}$$

Assume that there is a projection $P : \mathbb{C}^N \rightarrow \mathbb{C}^N$ such that

$$\hat{e} \in (I - P)(\mathbb{C}^N);\tag{19a}$$

Assume that there are a bounded function $h : [t_0, +\infty[\times [t_0, +\infty[\rightarrow [0, +\infty[$ and a constant $M > 0$ such that

$$\|\hat{Z}(t, s) P\| \leq h(t, s) |\tilde{e}(t, s)|,\tag{20a}$$

for all $t, s \in [t_0, +\infty[$ such that $t \geq s$;

$$\|\hat{Z}(t, s)\| \leq M |\tilde{e}(t, s)|;\tag{20b}$$

for all $t, s \in [t_0, +\infty[$ such that $t < s$.

Assume that

$$h(t, s) \rightarrow 0 \text{ as } t \rightarrow +\infty;\tag{20c}$$

$$h(t, s) \leq h(t, T)h(T, s) \text{ if } t \geq T \geq s. \quad (20d)$$

It will be used the variation of parameters formula for DEPCAG [3, 9] as follows.

Let's extend the conditions (20a) and (20b) as

$$|\hat{Z}(t, s)P| \leq M|\tilde{e}(t, s)|h(t, s), \text{ for } t \geq s, \quad (21)$$

and

$$|\hat{Z}(t, s)(I - P)| \leq M|\tilde{e}(t, s)|, \text{ for } t \leq s, \quad (22)$$

respectively.

Let $\Xi(t, s) = \tilde{e}(t, s)^{-1}\hat{Z}(t, s)$, for $t, s \geq 0$. Sea $G(t, s) : [0, +\infty[^2 \rightarrow \mathcal{M}_N(\mathbb{C})$ defined for

$$G(t, s) = \tilde{e}(t, s) \times \begin{cases} \Xi(t, s)P, & \text{if } t \geq s \\ -\Xi(t, s)(I - P), & \text{if } t < s. \end{cases}$$

Then, $\|G(t, s)\| \leq h(t, s)|\tilde{e}(t, s)| \leq K|\tilde{e}(t, s)|$, for all $t, s \geq 0$.

An additional condition on the perturbation $F(t, x(g(t)))$ is

$$\int_{t_0}^{+\infty} |\tilde{e}(t, g(t))|^{-1}\eta(t)dt < +\infty, \quad (23)$$

Let $n_0 \in \mathbb{N}_0$. Sea \mathcal{B}_{n_0} the set of functions $y : [n_0, +\infty[\rightarrow \mathbb{C}^N$ such that $\tilde{e}(\cdot, t_{n_0})^{-1}y \in L^\infty$. For $y \in \mathcal{B}_{n_0}$, sea $\|y\|_{n_0} = \sup_{t \geq t_{n_0}} |\tilde{e}(t, t_{n_0})|^{-1}|y(t)|$. Then,

$(\mathcal{B}_{n_0}, \|\cdot\|_{n_0})$ is Banach space, which is isometrically isomorphic to the Banach space $(L^\infty, \|\cdot\|_\infty)$.

Let $\mathcal{N} : \mathcal{B}_{n_0} \rightarrow \mathcal{B}_{n_0}$ be an operator defined by

$$(\mathcal{N}y)(t) = \tilde{e}(t, t_{n_0})\hat{e} + \int_{t_{n_0}}^{+\infty} G(t, s)F(s, g(s))ds, \quad (24)$$

for all $y \in \mathcal{B}_{n_0}$ and $t \geq t_{n_0}$.

Notice that the operator \mathcal{N} is well defined. In fact, let $y \in \mathcal{B}_{n_0}$. Then,

$$\begin{aligned} |\tilde{e}(t, t_{n_0})|^{-1}|(\mathcal{N}y)(t)| &\leq |\hat{e}| + M \int_{t_{n_0}}^t h(t, s) |\tilde{e}(s, g(s))^{-1}F(s, g(s))| |\tilde{e}(g(s), t_{n_0})^{-1}y(g(s))| ds \\ &\quad + K \int_t^\infty |\tilde{e}(s, g(s))^{-1}F(s, g(s))| |\tilde{e}(g(s), t_{n_0})^{-1}y(g(s))| ds \\ &\leq 1 + \Theta_{n_0}(t)\|y\|_{n_0}, \end{aligned}$$

where $\Theta_{n_0}(t) = M \int_{t_{n_0}}^t h(t, s) |\tilde{e}(s, g(s))|^{-1}\eta(s) ds + M \int_t^\infty |\tilde{e}(s, g(s))|^{-1}\eta(s) ds$.

By (23), $\Theta_{n_0}(t) \leq \sup_{\tau \geq t_{n_0}} \Theta_{n_0}(\tau) < +\infty$.

So, $\mathcal{N}y \in \mathcal{B}_{n_0}$.

Let n_0 so large that $\Theta_{n_0}(\infty) < 1$. Since

$$\|\mathcal{N}y_1 - \mathcal{N}y_2\|_{n_0} \leq \Theta_{n_0}(\infty)\|y_1 - y_2\|_{n_0},$$

for all $y_1, y_2 \in \mathcal{B}_{n_0}$, by the Banach Fixed Point Theorem, there is a only one $y \in \mathcal{B}_{n_0}$ such that $y = \mathcal{N}y$.

Moreover,

$$|\tilde{e}(t, t_{n_0})^{-1}y(t) - \hat{e}| \leq \Theta_{n_0}(t)\|y\|_{n_0}. \quad (25)$$

Due to the conditions (3) y (23), we have that

$$|\tilde{e}(\cdot, g(\cdot))|^{-1}\eta(\cdot)| \in L^1.$$

It is not hard to see that $\lim_{t \rightarrow +\infty} \Theta_{n_0}(t) = 0$. By considering

$$w(t) = \tilde{e}(t, t_{n_0})^{-1}y(t) - \hat{e},$$

by the inequality (25), we have $w(t) \rightarrow 0$ as $t \rightarrow +\infty$.

So, the constructed contractive operator allows us to state the following result.

Lemma 1 *Assume for that DEPCAG (4) the conditions (9a), (20c), (20d), (21), (22) are satisfied and the conditions (3) and (23) are satisfied for F . Then, the operator \mathcal{N} defined by (24) satisfies:*

1. $\mathcal{N}(\mathcal{B}_{n_0}) \subseteq \mathcal{B}_{n_0}$;
2. $\tilde{e}(t, n_0)^{-1}[(\mathcal{N}y)(t) - \hat{e}] \rightarrow 0$ as $t \rightarrow +\infty$;
3. \mathcal{N} is contractive for n_0 large enough;
4. \mathcal{N} has a only one fixed point $y_{n_0} \in \mathcal{B}_{n_0}$, i. e., $y_{n_0} = \mathcal{N}(y_{n_0})$;
5. The fixed point y_{n_0} satisfies the asymptotic formula

$$y_{n_0}(t) = \tilde{e}(t, n_0)(\hat{e} + w(t)), \quad (26)$$

where $w(t) \rightarrow 0$ as $t \rightarrow +\infty$.

3 Main Result

Now, we are in conditions to present our main result.

Theorem 2 *Assume that $\xi_n = t_n$ for all $n \in \mathbb{N}_0$ an that conditions (9a), (13a), (19a), (20c), (20d), (21), (22) and (23) are satisfied. Then, the DEPCAG (2) has a solution $y = y(t)$ defined for $t \geq t_{n_0}$ with n_0 large enough such that*

$$y(t) = \tilde{e}(t, t_{n_0})(\hat{e} + w(t)), \quad (27)$$

where $w(t) \rightarrow 0$ as $t \rightarrow +\infty$.

Proof: Since the conditions of Teorema 1 are satisfied, the operator \mathcal{N} defined by

$$(\mathcal{N}y)(t) = \tilde{e}(t, t_{n_0})\hat{e} + \int_{t_{n_0}}^{+\infty} G(t, s)F(s, y(g(s)))ds$$

has a fixed point $y = y(t)$ defined for all $t \geq t_{n_0}$ with n_0 large enough with the asymptotic formula (26) which takes the form (27).

From (17) it can be easily proved that y is solution of (2). Therefore the DEPCAG (2) has a solution $y = y(t)$ defined for $t \geq t_{n_0}$ with n_0 large enough with the asymptotic formula (27)

□

Example Consider the diagonal matrices in $\mathcal{M}_N(\mathbb{C})$,

$$\Lambda_A(t) = \text{diag}(a_1(t), a_2(t), \dots, a_N(t)) \text{ and } \Lambda_B(t) = \text{diag}(b_1(t), b_2(t), \dots, b_N(t))$$

DEPCAG with delayed piecewise constant argument $\xi_n = t_n$, for all $n \in \mathbb{N}_0$,

$$z'(t) = \Lambda_A(t)z(t) + \Lambda_B(t)z(\gamma(t)), \quad (28)$$

for $t \geq t_0$. Then, the solutions of (28) can be written as

$$y(t) = Z(t, s)y(s),$$

where $Z(t, s) = \text{diag}(e_1(t, s), e_2(t, s), \dots, e_N(t, s))$,

$$\begin{aligned} e_l(t, s) &= e^{\int_s^t a_l(\xi)d\xi} \left(1 + \int_{t_{\gamma(s)}}^s e^{-\int_{\gamma(s)}^\sigma a_l(\xi)d\xi} b_l(\sigma)d\sigma \right)^{-1} \\ &\times \left[\prod_{m=\gamma(s)}^{k_t-1} \left(1 + \int_{t_m}^{t_{m+1}} e^{-\int_{\gamma(s)}^\sigma a_l(\xi)d\xi} b_l(\sigma)d\sigma \right) \right] \\ &\times \left(1 + \int_{\gamma(t)}^t e^{-\int_{\gamma(t)}^\sigma a_l(\xi)d\xi} b_l(\sigma)d\sigma \right), \end{aligned}$$

for $l \in \{1, \dots, N\}$ y $t \geq s$.

We assume that

$$1 + \int_{t_m}^t e^{-\int_{\gamma(s)}^\sigma a_l(\xi)d\xi} b_l(\sigma)d\sigma \neq 0, \text{ for all } l \in \{1, \dots, N\}, t \in [t_m, t_{m+1}[\text{ y } m \in \mathbb{N}_0. \quad (29)$$

This is equivalent to (9a) and guarantees that $Z(t, s)$ is invertible for $t \geq s$ and the existence of $Z(t, s)$ for $t < s$. Moreover, $e_l(t, s) = \frac{1}{e_l(s, t)}$ for $t < s$.

The following is a direct application of theorem 2.

Corollary 1 Assume that there is $k \in \{1, \dots, N\}$ such that

$$(a) \quad \lim_{t \rightarrow +\infty} \left| \frac{e_l(t, s)}{e_k(t, s)} \right| = 0, \text{ for } l < k,$$

$$(b) \quad \text{There is } C \geq 0 \text{ such that } \left| \frac{e_l(t, s)}{e_k(t, s)} \right| \leq C, \text{ for } l \geq k.$$

Let $\{R(t)\}_{t \geq 0}$ a family in $\mathcal{M}_N(\mathbb{C})$ such that $R(\cdot)$ is locally integrable and

$$\sum_{n=n_0}^{+\infty} \int_{t_n}^{t_{n+1}} |e_k(s, t_n)^{-1}| \|R(s)\| ds \text{ is convergent.} \quad (30)$$

Then, the DEPCAG

$$y'(t) = \Lambda_A(t)y(t) + \Lambda_B(t)y(\gamma(t)) + R(t)y(\gamma(t)),$$

has a solution $y = y(t)$ defined for $t \geq t_{n_0}$ with n_0 large enough such that

$$y(t) = e_k(t, t_{n_0})(e_k + w(t)), \quad (31)$$

where e_k is the k -th vector of the canonical base of \mathbb{C}^N and $w(t) \rightarrow 0$ as $t \rightarrow +\infty$.

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